1. Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ of the space curves below.

   (a) (18 points) $\alpha(s) = \left(\frac{4}{9}(1+s)^{\frac{3}{2}}, \frac{4}{9}(1-s)^{\frac{3}{2}}, \frac{1}{3}s\right), s \in (-1,1)$.

   (b) (12 points) $\alpha(\theta) = \left(6\cos 2\theta \cos^3\left(\frac{2\theta}{3}\right),\ 6\sin 2\theta \cos^3\left(\frac{2\theta}{3}\right),\ \frac{1}{2}\cos 4\theta - \cos^2 2\theta\right),\ \theta \in \left(0,\ \frac{\pi}{4}\right)$.

Solution. (By Max Shung)

(a) Observe that

$$\alpha'(s) = \left(\frac{2}{3}(1+s)^{\frac{1}{2}}, -\frac{2}{3}(1-s)^{\frac{1}{2}}, \frac{1}{3}\right)$$

and

$$\|\alpha'(s)\|^2 = \frac{4}{9}(1+s) + \frac{4}{9}(1-s) + \frac{1}{9} = 1$$

Hence, α is an arc-length parametrization.
Then, the unit tangent vector is

$$\mathbf{T}(s) = \alpha'(s) = \left(\frac{2}{3}(1+s)^{\frac{1}{2}}, -\frac{2}{3}(1-s)^{\frac{1}{2}}, \frac{1}{3}\right)$$

$$\mathbf{T}'(s) = \left(\frac{1}{3}(1+s)^{-\frac{1}{2}}, \frac{1}{3}(1-s)^{-\frac{1}{2}}, 0\right)$$

and

$$\|\mathbf{T}'(s)\| = \sqrt{\frac{1}{9(1+s)} + \frac{1}{9(1-s)}} = \frac{1}{3}\sqrt{\frac{2}{1-s^2}}$$

Therefore, the curvature is

$$\kappa(s) = \|\mathbf{T}'(s)\| = \frac{1}{3}\sqrt{\frac{2}{1-s^2}}$$

and $\kappa(s) > 0$ for all $s \in (-1,1)$
The unit normal vector is given by

$$\mathbf{N}(s) = \frac{1}{\kappa(s)}\mathbf{T}'(s) = \left(\frac{1}{\sqrt{2}}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}(1+s)^{\frac{1}{2}}, 0\right)$$

The binormal vector is given by

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \left(-\frac{1}{3\sqrt{2}}(1+s)^{\frac{1}{2}}, \frac{1}{3\sqrt{2}}(1-s)^{\frac{1}{2}}, \frac{2\sqrt{2}}{3}\right)$$

Differentiating $\mathbf{N}(s)$ with respect to s, we have

$$\mathbf{N}'(s) = \left(-\frac{1}{2\sqrt{2}}(1-s)^{-\frac{1}{2}}, \frac{1}{2\sqrt{2}}(1+s)^{-\frac{1}{2}}, 0\right)$$

and thus the torsion is given by

$$\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle = \frac{1}{12}\left(\sqrt{\frac{1+s}{1-s}} + \sqrt{\frac{1-s}{1+s}}\right) = \frac{1}{6\sqrt{1-s^2}}.$$

Remark. Many of you missed the minus sign for $\mathbf{j}$-component of $\alpha'(s)$ carelessly. It leads you getting wrong $\mathbf{T}, \mathbf{N}, \mathbf{B}$ and torsion $\tau(s)$, so the mark deduction will be serious. I try to give you marks for Q1(a) as many as I can, please do more practice on the computational problems for preparation for final exam!

(b) First, we compute the differentiation

$$\begin{aligned}\frac{d}{d\theta}\left[6\cos 2\theta\cos^3\left(\frac{2\theta}{3}\right)\right] &= -12\cos 2\theta\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{2\theta}{3}\right) - 12\sin 2\theta\cos^3\left(\frac{2\theta}{3}\right)\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\left[\cos 2\theta\sin\left(\frac{2\theta}{3}\right) + \sin 2\theta\cos\left(\frac{2\theta}{3}\right)\right]\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\left(\frac{2\theta}{3}\right) + 2\theta\right)\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{8\theta}{3}\right)\end{aligned}$$

Similarly, we also have

$$\frac{d}{d\theta}\left[6\sin 2\theta\cos^3\left(\frac{2\theta}{3}\right)\right] = 12\cos^2\left(\frac{2\theta}{3}\right)\cos\left(\frac{8\theta}{3}\right)$$

Therefore, we have $\alpha'(\theta) = \left(-12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{8\theta}{3}\right), 12\cos^2\left(\frac{2\theta}{3}\right)\cos\left(\frac{8\theta}{3}\right), 0\right)$ and

$$\|\alpha'(\theta)\|^2 = 144\cos^4\left(\frac{2\theta}{3}\right) \implies \|\alpha'(\theta)\| = 12\cos^2\left(\frac{2\theta}{3}\right)$$

Hence, the unit tangent vector is

$$\mathbf{T}(\theta) = \left(-\sin\left(\frac{8\theta}{3}\right), \cos\left(\frac{8\theta}{3}\right), 0\right)$$

and

$$\mathbf{T}'(\theta) = \left(-\frac{8}{3}\cos\left(\frac{8\theta}{3}\right), -\frac{8}{3}\sin\left(\frac{8\theta}{3}\right), 0\right)$$

Therefore, the curvature is given by

$$\kappa(\theta) = \frac{\|\mathbf{T}'(\theta)\|}{\|\alpha'(\theta)\|} = \frac{\frac{8}{3}}{12\cos^2\frac{2\theta}{3}} = \frac{2}{9\cos^2\frac{2\theta}{3}}$$

Also, the unit normal vector is

$$\mathbf{N}(\theta) = \left(-\cos\left(\frac{8\theta}{3}\right), -\sin\left(\frac{8\theta}{3}\right), 0\right)$$

Observe that

$$\mathbf{T}(\theta) \times \mathbf{N}(\theta) = (0, 0, 1)$$

and it is a unit vector.
Therefore, the unit binormal vector is

$$\mathbf{B} = (0, 0, 1)$$

and the torsion is given by

$$\tau = \langle \mathbf{N}'(\theta), \mathbf{B}\rangle = 0.$$

Remark. Almost all of you failed to figure out $\alpha'(\theta)$ in part (b) and many of you did not simplify $\alpha'(\theta)$ first and directly applied differentiation to get $\alpha''(\theta)$. To be honest, it is the simplest way to kill yourself in part (b), not really recommend you to do so. My marking on Q1(b) is harsh and many of you get 0 or 1 point in this part.

2. Let $\mathbf{r} : (0, \ln 2) \to \mathbb{R}^3$ be a curve defined by $\mathbf{r}(t) = (\cosh t, \sinh t, t)$.

   (a) (2 points) Suppose $x \in \mathbb{R}$. Show that there exists exactly one $t \in \mathbb{R}$ such that $\sinh t = x$, and express t in terms of x.

   (b) (10 points) Let $\mathbf{r}_1(s)$ be the arc-length parameterization of the curve. Find an explicit formula for $\mathbf{r}_1(s)$. State clearly the range of possible values of the parameter s.

   (c) (4 points) Compute the tangent vector $\mathbf{T}(s_0)$ of $\mathbf{r}_1(s)$ at $s = s_0$.

Solution. (By Max Wong)

(a)

$$x = \sinh t = \frac{e^t - e^{-t}}{2}$$
$$\left(e^t\right)^2 - 2xe^t - 1 = 0$$
$$\left(e^t - x\right)^2 - x^2 - 1 = 0$$

Solving gives

$$e^t = x \pm \sqrt{x^2+1}$$

Note that $x - \sqrt{x^2+1} < 0$ for any $x \in \mathbb{R}$. Then $e^t = x + \sqrt{x^2+1}$. Taking ln on both sides gives $t = \ln\left(x + \sqrt{x^2+1}\right)$. Therefore $t = \ln\left(x + \sqrt{x^2+1}\right)$ is the only real number such that $\sinh t = x$.

(b) For any $t \in (0, \ln 2)$,

$$s(t) = \int_0^t \|\mathbf{r}'(t)\|\, dt = \int_0^t \sqrt{\sinh^2 t + \cosh^2 t + 1}\, dt = \int_0^t \sqrt{2\cosh^2 t}\, dt = \sqrt{2}\sinh t$$

Therefore, $\sinh t = \dfrac{s}{\sqrt{2}}$. Using the result from (a),

$$t = \ln\left(\frac{s}{\sqrt{2}} + \sqrt{\left(\frac{s}{\sqrt{2}}\right)^2 + 1}\right) = \ln\left(s + \sqrt{s^2+2}\right) - \ln\sqrt{2}$$

Therefore, $\mathbf{r}_1(s) \overset{\text{def}}{=} \mathbf{r}\left(\ln\left(s + \sqrt{s^2+2}\right) - \ln\sqrt{2}\right)$ is an arc-length parameterization of the curve. Simplifying gives $\mathbf{r}_1(s) = \left(\dfrac{\sqrt{2(s^2+2)}}{2}, \dfrac{s\sqrt{2}}{2}, \ln\left(s + \sqrt{s^2+2}\right) - \ln\sqrt{2}\right)$. Note that $s(t)$ is strictly increasing. Also, $\lim\limits_{t \to (\ln 2)^-} s(t) = \sqrt{2}\sinh(\ln 2) = \dfrac{3}{4}\sqrt{2}$. Then $s \in \left(0, \dfrac{3}{4}\sqrt{2}\right)$.

(c) Note that $\|\mathbf{r}_1'(s)\| = 1$ for any $s \in \left(0, \dfrac{3}{4}\sqrt{2}\right)$. Then,

$$\mathbf{T}(s_0) = \frac{\mathbf{r}_1'(s_0)}{\|\mathbf{r}_1'(s_0)\|} = \mathbf{r}_1'(s_0) = \left(\frac{\sqrt{2}}{2} \cdot \frac{2s_0}{2\sqrt{s_0^2+2}}, \frac{\sqrt{2}}{2}, \frac{1 + \dfrac{2s_0}{2\sqrt{s_0^2+2}}}{s_0 + \sqrt{s_0^2+2}}\right) = \left(\frac{\sqrt{2}s_0}{2\sqrt{s_0^2+2}}, \frac{\sqrt{2}}{2}, \frac{1}{\sqrt{s_0^2+2}}\right)$$

3. The logarithmic spiral is a curve defined by $r = e^\theta$ in polar coordiantes.
   (a) (6 points) Find the arc-length of the logarithmic spiral from $\theta = 0$ to $\theta = 2\pi$.
   (b) (8 points) Find the curvature of the logarithmic spiral.

Solution. (By Alex Tang)

(a) The parameterization of the curve is given by

$$\mathbf{x}(\theta) = (r\cos\theta, r\sin\theta) = (e^\theta\cos\theta, e^\theta\sin\theta) \text{ for any } \theta \in \mathbb{R}$$

Then, by product rule, we have

$$\mathbf{x}'(\theta) = (e^\theta\cos\theta - e^\theta\sin\theta, e^\theta\sin\theta + e^\theta\cos\theta) = e^\theta(\cos\theta - \sin\theta, \cos\theta + \sin\theta)$$

Taking norm, yield

$$\|\mathbf{x}'(\theta)\| = e^\theta\sqrt{(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2} = e^\theta\sqrt{2}$$

Therefore, the arc-length from $\theta = 0$ to $\theta = 2\pi$ is given by

$$\int_0^{2\pi} \|\mathbf{x}'(\theta)\| \, \mathrm{d}\theta = \int_0^{2\pi} e^\theta\sqrt{2} \, \mathrm{d}\theta = \sqrt{2}(e^{2\pi} - 1)$$

(b) The unit tangent vector $\mathbf{T}$ is given by

$$\mathbf{T}(\theta) = \frac{\mathbf{x}'(\theta)}{\|\mathbf{x}'(\theta)\|} = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta, \cos\theta + \sin\theta)$$

We also have

$$\mathbf{T}'(\theta) = \frac{1}{\sqrt{2}}(-\cos\theta - \sin\theta, \cos\theta - \sin\theta)$$

Taking norm, yield

$$\|\mathbf{T}'(\theta)\| = \frac{1}{\sqrt{2}}(\sqrt{2}) = 1$$

Therefore, the required curvature κ is given by

$$\kappa(\theta) = \frac{\|\mathbf{T}'(\theta)\|}{\|\mathbf{x}'(\theta)\|} = \frac{1}{e^\theta\sqrt{2}} = \frac{\sqrt{2}}{2}e^{-\theta}$$

Alternative solution:

(a) Recall the arc-length formula for polar coordinates is given by

$$\int_I \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\,\mathrm{d}\theta$$

In this question, $r(\theta) = e^\theta$, and $r'(\theta) = e^\theta$ Thus, the arc-length from $\theta = 0$ to $\theta = 2\pi$ is given by

$$\int_0^{2\pi} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\,\mathrm{d}\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\,\mathrm{d}\theta = \int_0^{2\pi} \sqrt{2}e^\theta\,\mathrm{d}\theta = \sqrt{2}(e^{2\pi} - 1)$$

(b) The parameterization of the curve is given by

$$\mathbf{x}(\theta) = (e^\theta \cos\theta, e^\theta \sin\theta) \text{ for any } \theta \in \mathbb{R}$$

Then, by product rule, we have

$$\mathbf{x}'(\theta) = (e^\theta \cos\theta - e^\theta \sin\theta, e^\theta \sin\theta + e^\theta \cos\theta) = e^\theta(\cos\theta - \sin\theta, \cos\theta + \sin\theta)$$

Again, by product rule, we have

$$\mathbf{x}''(\theta) = (e^\theta \cos\theta - e^\theta \sin\theta - e^\theta \sin\theta - e^\theta \cos\theta, e^\theta \sin\theta + e^\theta \cos\theta + e^\theta \cos\theta - e^\theta \sin\theta) = (-2e^\theta \sin\theta, 2e^\theta \cos\theta)$$

Taking norm of $\mathbf{x}'$, yield

$$\|\mathbf{x}'(\theta)\| = e^\theta \sqrt{(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2} = e^\theta \sqrt{2}$$

Then, using $\kappa = \dfrac{det(\mathbf{x}', \mathbf{x}'')}{\|\mathbf{x}'\|^3}$, we have

$$\kappa(\theta) = \frac{(2e^{2\theta}(\cos^2\theta - \sin\theta\cos\theta) + 2e^{2\theta}(\sin\cos\theta + \sin^2\theta)}{2\sqrt{2}e^{3\theta}} = \frac{\sqrt{2}}{2}e^{-\theta}$$

4. (15 points) Let $\mathbf{r}(t)$ be a regular parametrized space curve with $\kappa(t) > 0$ for any t. Denote the torsion at $\mathbf{r}(t)$ by $\tau(t)$. Prove that $\mathbf{r}(t)$ is contained in a plane if and only if $\tau(t) = 0$ for any t.
(Hint: A space curve $\mathbf{r}$ is contained in a plane if there exists a fixed unit vector $\mathbf{n}$ such that $\langle \mathbf{r}, \mathbf{n}\rangle$ is a constant.)

Solution. (By Michael Cheung)

Suppose $\mathbf{r}$ is a plane curve. For convenience, we may consider the curve's arc length parametrization $\mathbf{r}$(s). Choose any fixed point x_0 on $\mathbf{r}$(s), there exists a constant unit normal vector $\mathbf{n}$ such that

$$\mathbf{r}(s) - x_0, \mathbf{n}\rangle = 0, \ \textit{i.e.}\ \langle \mathbf{r}(s), \mathbf{n}\rangle = \langle x_0, \mathbf{n}\rangle = a$$

which is a constant. Notice that

$$\begin{cases} \langle \mathbf{r}'(s), \mathbf{n}\rangle = \langle \mathbf{r}'(s), \mathbf{n}\rangle + \langle \mathbf{r}(s), \mathbf{n}'\rangle = \dfrac{d}{ds}\langle \mathbf{r}(s), \mathbf{n}\rangle = \dfrac{d}{ds}a = 0 \\ \langle \mathbf{r}''(s), \mathbf{n}\rangle = \langle \mathbf{r}''(s), \mathbf{n}\rangle + \langle \mathbf{r}'(s), \mathbf{n}'\rangle = \dfrac{d}{ds}\langle \mathbf{r}'(s), \mathbf{n}\rangle = 0 \end{cases}$$

From the first equation, we have $\langle \mathbf{T}, \mathbf{n}\rangle = \langle \mathbf{r}'(s), \mathbf{n}\rangle = 0$.
From the second equation, we have $\langle \kappa\mathbf{N}, \mathbf{n}\rangle = \kappa\langle \mathbf{N}, \mathbf{n}\rangle = 0$, i.e. $\langle \mathbf{N}, \mathbf{n}\rangle = 0$ as $\kappa > 0$
Since $\mathbf{B}$ is a unit vector, we have $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \pm\mathbf{n}$, which is a constant vector.
Hence $\mathbf{B}' = -\tau\mathbf{N} = \mathbf{0}$, *i.e.* $\tau \equiv 0$ *as* $\mathbf{N}$ is a non-zero vector.

Conversely, WLOG, suppose $\tau(s) = 0$ for any s. Then we have $\mathbf{B}' = -\tau\mathbf{N} = \mathbf{0}$.
Hence $\mathbf{B}$ is a constant vector and $\dfrac{d}{ds}\langle \mathbf{r}, \mathbf{B}\rangle = \langle \mathbf{r}', \mathbf{B}\rangle + \langle \mathbf{r}, \mathbf{B}'\rangle = \langle \mathbf{T}, \mathbf{B}\rangle - \tau\langle \mathbf{r}, \mathbf{N}\rangle = 0$.
Therefore $\langle \mathbf{r}, \mathbf{B}\rangle$ is a constant, which implies $\mathbf{r}$ is a plane curve lying on a plane with normal vector $\mathbf{B}$.

OR

Conversely, WLOG, suppose $\tau(s) \equiv 0$, then $\mathbf{B}' = -\tau\mathbf{N} \equiv \mathbf{0}$.
Hence $\mathbf{B}$ is a constant vector.
Pick arbitrary point $\mathbf{r}(s_0)$ on $\mathbf{r}$. Consider the plane $\mathbf{f}(s) = \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{B}\rangle$, we have

$$\mathbf{f}'(s) = \langle \mathbf{r}'(s), \mathbf{B}\rangle + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{B}'\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = 0$$

Hence $\mathbf{f}(s)$ is a constant, and $\mathbf{f}(s_0) = \langle \mathbf{r}(s_0) - \mathbf{r}(s_0), \mathbf{B}\rangle = 0$, which implies $\mathbf{f}(s) \equiv 0$
Therefore $\mathbf{r}$ is a plane curve.

Remark. This question is directly copied from Proposition 2.4.6, p.90 on lecture notes. I understand that the DSE syllabus makes you feel like proofs are not important in Mathematics, but this is a *blunder*! If you cannot answer this question, you are not stupid, you are just lazy and lack preparation. Hope to see your well-prepared performance in your final exam.

5. (12 points) Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular arc length parametrized plane curve. Suppose that $\mathbf{p} \in \mathbb{R}^2$ is a point such that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$. Suppose there exists $s_0 \in I$ such that

$$\|\alpha(s_0) - \mathbf{p}\| = \max_{s\in I}\{\|\alpha(s) - \mathbf{p}\|\}.$$

Denote the curvature of α at $s = s_0$ by $\kappa(s_0)$. Show that

$$|\kappa(s_0)| \geq \min_{s\in I}\left\{\frac{1}{\|\alpha(s) - \mathbf{p}\|}\right\}.$$

Proof. (By Max Shung)
First, note that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$, hence $\|\alpha(s) - \mathbf{p}\|$ is differentiable on I.
Then, we consider

$$\frac{d}{ds}\|\alpha(s) - \mathbf{p}\| = \frac{\langle \alpha'(s), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|}$$
$$\frac{d^2}{ds^2}\|\alpha(s) - \mathbf{p}\| = \frac{\langle \alpha''(s), \alpha(s) - \mathbf{p}\rangle + \langle \alpha'(s), \alpha'(s)\rangle}{\|\alpha(s) - \mathbf{p}\|} - \frac{\langle \alpha'(s), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|^2}\frac{d}{ds}\|\alpha(s) - \mathbf{p}\|$$

Since $\|\alpha(s) - \mathbf{p}\|$ attains maximum at $s = s_0 \in I$, hence it follows that

$$\left.\frac{d}{ds}\right|_{s=s_0}\|\alpha(s) - \mathbf{p}\| = 0 \implies \frac{\langle \alpha'(s_0), \alpha(s_0) - \mathbf{p}\rangle}{\|\alpha(s_0) - \mathbf{p}\|} = 0$$

and

$$\left.\frac{d^2}{ds^2}\right|_{s=s_0}\|\alpha(s) - \mathbf{p}\| \leq 0 \implies \frac{\langle \alpha''(s_0), \alpha(s_0) - \mathbf{p}\rangle + 1}{\|\alpha(s_0) - \mathbf{p}\|} \leq 0$$

as $\langle \alpha'(s), \alpha'(s)\rangle = 1$ for any $s \in I$.
Observe that

$$\frac{\langle \alpha'(s_0), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|} = 0, \ \frac{\alpha(s_0) - \mathbf{p}}{\|\alpha(s_0) - \mathbf{p}\|} = \pm\mathbf{N}(s_0),$$

where $\mathbf{N}(s_0)$ denotes the unit normal vector to α at $s = s_0$.
Therefore, it follows that

$$\begin{aligned}
\langle \alpha''(s_0), \pm\mathbf{N}(s_0)\rangle + \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} &\leq 0 \\
-\langle \kappa(s_0)\mathbf{N}(s_0), \pm\mathbf{N}(s_0)\rangle &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \\
\mp\kappa(s_0)\cdot 1 &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \\
|\kappa(s_0)| &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} = \frac{1}{\max_{s\in I}\|\alpha(s) - \mathbf{p}\|} \qquad ...(*)
\end{aligned}$$

As $\alpha(s) \neq \mathbf{p}$ for all $s \in I$, hence $\|\alpha(s) - \mathbf{p}\| > 0$ and it is bounded above by $\|\alpha(s_0) - \mathbf{p}\|$, it follows that

$$\frac{1}{\|\alpha(s) - \mathbf{p}\|} \geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \qquad \forall s \in I$$

Therefore, by definition of minimal element of a set, we have

$$\min_{s\in I}\left\{\frac{1}{\|\alpha(s) - \mathbf{p}\|}\right\} = \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} = \frac{1}{\max_{s\in I}\|\alpha(s) - \mathbf{p}\|}$$

and putting back to (*) and thus the result follows.
Remark. Almost all of you cannot state the differentiablility for norm function, and use first and second differentiation to carry forward. Furthermore, none of you can correctly prove that the relation of the reciprocal of a maximum and the minimum of the reciprocal as shown above. The performance for this question is expected because I am a killer! – *Max Shung*

6. Let $\{\gamma_t(u)\}$ be a family of closed smooth plane curves with $u \in [0, 2\pi)$ and $t \in I$ where I is open interval. Let $\gamma_t(s)$ be the arc-length parametrization of $\gamma_t(u)$ for each t. Suppose it satisfies the following heat equation (with notation $\gamma = \gamma(s,t) = \gamma_t(s)$):

$$\frac{\partial}{\partial t}\gamma = \frac{\partial^2}{\partial s^2}\gamma.$$

(a) (3 points) Explain why

$$\frac{\mathrm{d}s}{\mathrm{d}u} = |\gamma'(u)|.$$

By inverse function theorem we therefore have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{|\gamma'(u)|}.$$

(b) (4 points) Given $\frac{\partial}{\partial t}|\gamma'(u)| = -\kappa^2|\gamma'(u)|$, show that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}f - \frac{\partial}{\partial s}\frac{\partial}{\partial t}f = \kappa^2\frac{\partial}{\partial s}f$$

for all smooth function $f(u,t)$.

(**Hint**: Start with $\dfrac{\partial}{\partial t}\dfrac{\partial f}{\partial s} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial s}\dfrac{\partial f}{\partial u}\right)$)

(c) i. (2 points) Do you agree that $\frac{\partial \mathbf{N}}{\partial t}$ must be perpendicular to $\mathbf{N}$? Explain.

ii. (4 points) Combine all the above information, show for the Frenet frame $\{\mathbf{T}, \mathbf{N}\}$ of $\gamma_t(s)$:

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \kappa}{\partial s}\mathbf{N}, \quad \frac{\partial \mathbf{N}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{T}.$$

Solution. (By Clive Chan)

(a) s is arclength.

(b) Let $v = |\gamma'(u)|$. By chain rule, $\partial_t\partial_s f = \partial_t(\frac{1}{v}\partial_u f) = -\frac{-\kappa^2 v}{v^2}\partial_u f + \frac{1}{v}\partial_u\partial_t f = \kappa^2\partial_s f + \partial_s\partial_t f$.

c(i) $\langle \mathbf{N}, \mathbf{N}\rangle = 1$ and differentiate both sides.

c(ii) $\partial_t\mathbf{T} = \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + \kappa^2\partial_s\gamma = \partial_s^3\gamma + \kappa^2\mathbf{T} = \partial_s(\kappa\mathbf{N}) + \kappa^2\mathbf{T} = (\partial_s\kappa)\mathbf{N}$ by Frenet frame.
$0 = \partial_t\langle \mathbf{T}, \mathbf{N}\rangle = \langle \partial_t\mathbf{T}, \mathbf{N}\rangle + \langle \mathbf{T}, \partial_t\mathbf{N}\rangle = \partial_s\kappa + \langle \mathbf{T}, \partial_t\mathbf{N}\rangle$ so $\partial_t\mathbf{N} = -(\partial_s\kappa)\mathbf{T}$ by part c(i).

Remark: Please revise chain rule, product rule and quotient rule if you find this difficult.

7. (15 points (bonus)) This question is related to the brachistochrone problem. It examines students' ability to read (easy) mathematical text in modern geometry.

   Consider the following function J mapping a function L to a scalar quantity:

$$J[L] = \int_{x_i}^{x_f} L(x, y(x), y'(x))\, dx$$

   where $x \in [x_i, x_f], y : [x_i, x_f] \to \mathbb{R}$ and L is a function on $x, y(x)$ and $y'(x)$.

   Since J is a function on functions, extreme points of J are actually curves (as points in a function space). Suppose J attain its minimum with the curve $y = y_{\min}(x)$.

   We can find $y_{\min}(x)$ by the following steps:

   1. Denote the neighboring curve of $y_{\min}(x)$ by

$$y = y_{\min}(x) + \alpha\eta(x)$$

      where α is a parameter and $\eta(x)$ is a function satisfies $\eta(x_i) = \eta(x_f) = 0$. (Refer to Figure 1.)

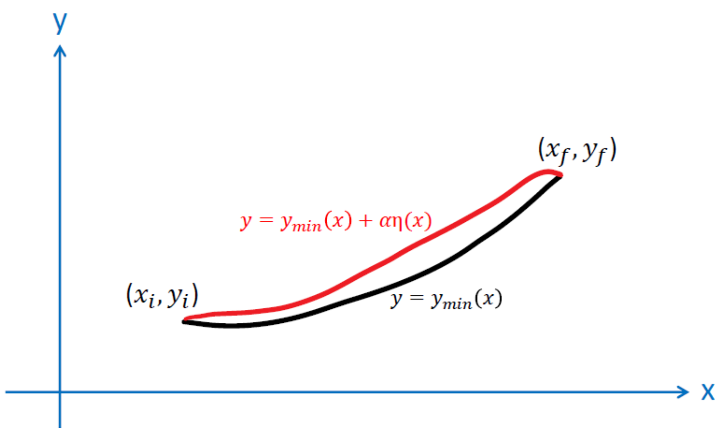

Figure 1: Illustration of Step 1

   2. We then have the relationship:

$$\begin{cases} y &= y_{\min}(x) + \alpha\eta(x) \\ y' &= y'_{\min}(x) + \alpha\eta'(x). \end{cases}$$

   3. Now we have

$$J = J(\alpha) = \int_{x_i}^{x_f} L(x, y_{\min}(x) + \alpha\eta(x), y'_{\min}(x) + \alpha\eta'(x))\mathrm{d}x.$$

      and the minimum of J occurs at $\alpha = 0$ (i.e. occurs at $y = y_{\min}(x)$). We also have

$$\frac{dJ}{d\alpha} = 0.$$

The questions are on the next page.

(a) i. It is given that $\dfrac{d}{d\alpha}\displaystyle\int_a^b h(x,\alpha)\,dx = \int_a^b \dfrac{\partial}{\partial\alpha} h(x,\alpha)\,dx$. Show that (3 marks)

$$\frac{dJ}{d\alpha} = \int_{x_i}^{x_f} \left(\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x) \right) dx$$

ii. Hence or otherwise, show that when $\dfrac{dJ}{d\alpha} = 0$, we have (3 marks)

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

(**Hint**: If $\int_{\mathbb{R}} f\eta = 0$ for all function η with compact support in $\mathbb{R}$, then $f = 0$.)

iii. Further suppose $\dfrac{\partial L}{\partial x} = 0$, show that if $\dfrac{dJ}{d\alpha} = 0$, then there exists $C \in \mathbb{R}$ such that (3 marks)

$$y'\frac{\partial L}{\partial y'} - L = C$$

(**Hint**: Differentiate $L(x, y(x), y'(x))$ with respect to x by multivariable chain rule)

(b) Clive would like to ride on a slide $\mathbf{r} : [0, a] \to \mathbb{R}^2$ in a playground, which is defined by

$$\mathbf{r}(x(t)) = (x(t), -y(x(t)))$$

where $y : [0, a] \to \mathbb{R}$ satisfies initial conditions $y(0) = 0, y(a) = b$.
Let $\mathbf{g}$ be a real constant, the time required for Clive to finish a ride is given by

$$T = \int_0^a \sqrt{\frac{1 + [y'(x)]^2}{2\mathbf{g}y(x)}}\,dx$$

i. Let $a, c \in \mathbb{R}^+$ with $c > a$, evaluate the integral (2 marks)

$$\int_0^a \sqrt{\frac{y}{c-y}}\,dy$$

ii. Using (a), show that the time minimising curve is parameterised by (3 marks)

$$\mathbf{y}_{\min}(\theta) = (A(\theta - \sin\theta), A(1 - \cos\theta))$$

for some constant A.
(You are not required to compute A and state the exact range of θ)

iii. Name the curve in (b)(ii). (1 mark)

Remark. This delightful question comes from Nelson and Tommy, Clive is still on his way sliding down.

Solution. (By Nelson Lam)

(a) Since it is given that $\frac{d}{d\alpha}\int_a^b h(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha}h(x,\alpha)\,dx$, by multivariable chain rule:

$$\begin{aligned}
J(\alpha) &\stackrel{\text{def}}{=} \int_{x_i}^{x_f} L(x, y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x))\,dx \\
J'(\alpha) &= \int_{x_i}^{x_f} \frac{\partial}{\partial\alpha} L(x, y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x))\,dx \\
&= \int_{x_i}^{x_f} \left[\frac{\partial L}{\partial x}\frac{\partial x}{\partial\alpha} + \frac{\partial L}{\partial y}\frac{\partial(y(x)+\alpha\eta(x))}{\partial\alpha} + \frac{\partial L}{\partial y'}\frac{\partial(y'(x)+\alpha\eta'(x))}{\partial\alpha}\right] dx \\
&= \int_{x_i}^{x_f} \left[0 + \frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x)\right] dx
\end{aligned}$$

Hence if $\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0$, using integration by part:

$$\begin{aligned}
\int_{x_i}^{x_f} \left[\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x)\right] dx &= 0 \\
\int_{x_i}^{x_f} \frac{\partial L}{\partial y}\eta(x)\,dx + \int_{x_i}^{x_f} \frac{\partial L}{\partial y'}\,d[\eta(x)] &= 0 \\
\int_{x_i}^{x_f} \frac{\partial L}{\partial y}\eta(x)\,dx + \left[\frac{\partial L}{\partial y'}\eta(x)\right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \eta(x)\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) dx &= 0 \\
\int_{x_i}^{x_f} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\,dx + \left[\frac{\partial L}{\partial y'}\eta(x)\right]_{x_i}^{x_f} &= 0
\end{aligned}$$

Given that $\eta(x_i) = \eta(x_f) = 0$ (vanish at the boundary), thus $\left[\frac{\partial L}{\partial y'}\eta(x)\right]_{x_i}^{x_f} = 0$

Also notice that $[x_i, x_f]$ is a compact set in $\mathbb{R}$, which supports the arbitrarily picken $\eta(x)$, hence:

$$\int_{x_i}^{x_f} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\,dx = 0, \forall\eta$$

Using **Hint**, we have $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

Further suppose $\frac{\partial L}{\partial x} = 0$ and $\frac{dJ}{d\alpha} = 0$, consider $\frac{d}{dx}L(x, y(x), y'(x)) = 0 + \frac{\partial L}{\partial y}y'(x) + \frac{\partial L}{\partial y'}y''(x)$

Multiplying $-y'(x)$ at both sides of $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

$$\begin{aligned}
y'(x)\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - y'(x)\frac{\partial L}{\partial y} &= 0 \\
y''(x)\frac{\partial L}{\partial y'} + y'(x)\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] - \left[y'(x)\frac{\partial L}{\partial y} + y''(x)\frac{\partial L}{\partial y'}\right] &= 0 \\
y''(x)\frac{\partial L}{\partial y'} + y'(x)\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] - \frac{dL}{dx} &= 0 \\
\frac{d}{dx}\left(y'\frac{\partial L}{\partial y'} - L\right) &= 0
\end{aligned}$$

By Fundamental Theorem of Calculus, there exists $C \in \mathbb{R}$ such that $y'\frac{\partial L}{\partial y'} - L = C$

(b) Pick $L(x, y(x), y'(x)) = \sqrt{\dfrac{1+[y'(x)]^2}{2\mathbf{g}y(x)}}$, $x_i = 0, x_f = a$, notice that $\dfrac{\partial L}{\partial x} = 0$

Hence there exists $C \in \mathbb{R}$ such that $y'\dfrac{\partial L}{\partial y'} - L = C$

By computation, $-\dfrac{1}{C} = \sqrt{(2\mathbf{g}y(x))\,(1+[y'(x)]^2)}$, that is: $y'(x) = \sqrt{\dfrac{c-y(x)}{y(x)}}$, where $c = \dfrac{1}{2gC^2}$

By Inverse Function Theorem, locally $x'(y) = \sqrt{\dfrac{y}{c-y}}$

By Fundamental Theorem of Calculus, $x = \displaystyle\int \sqrt{\frac{y}{c-y}}\,dy$

$$\begin{aligned}
x &= \int_0^a \sqrt{\frac{y}{c-y}}\,dy \\
&= \int_0^{\sin^{-1}(\sqrt{a/c})} \sqrt{\frac{c\sin^2\theta}{c-c\sin^2\theta}}2c\sin\theta\cos\theta\,d\theta \qquad [\text{Using the substitution } y = c\sin^2\theta] \\
&= 2c\int_0^{\sin^{-1}(\sqrt{a/c})} \sin^2\theta\,d\theta \\
&= c\int_0^{\sin^{-1}(\sqrt{a/c})} (1-\cos 2\theta)\,d\theta \\
&= \left[\frac{c}{2}(2\theta - \sin 2\theta)\right]_0^{\sin^{-1}(\sqrt{a/c})} \\
&= \frac{c}{2}\left[2\sin^{-1}(\sqrt{a/c}) - \sin^2\left(2\sin^{-1}(\sqrt{a/c})\right)\right] \\
&= \frac{c}{2}\left[2\sin^{-1}(\sqrt{a/c}) - \frac{4a(c-a)}{c^2}\right]
\end{aligned}$$

Denote $\theta(z) = \sin^{-1}(\sqrt{z/c})$, notice that $\dfrac{\partial\theta}{\partial z} = \dfrac{1}{2\sqrt{z(c-z)}} > 0, \forall z \in [0, a]$

Hence reparameterize x by θ, instead of z, we have $x = \dfrac{c}{2}(2\theta - \sin 2\theta)$, $y = c\sin^2\theta = \dfrac{c}{2}(1-\cos 2\theta)$

Normalizing gives: $\mathbf{y}_{\min}(\theta) = (A(\theta - \sin\theta), A(1-\cos\theta))$

Remark. There are 12 effective attempts on the question. Admittedly, this is a controversial and challenging question. Yet a lot of hidden clues (and explicit hints) are included to assist candidates to attempt the question. Working backward, trying to guess what happened in the previous step and hence establishing the logical linkage are techniques tested. Speaking of each part:

(a) (i). A number of you successfully identified $h(x, \alpha) = L(x, f(x) + \alpha\eta(x), f'(x) + \alpha\eta'(x))$ and applied Leibniz Integral Rule. Although some of you are unfamiliar with multivariable chain rule. It is still glad to see many of you try to attempt it.

(a) (ii). Honourable Mention: One candidate successfully applied integration by part and following **Hint**.

To approach to this question: Following **Hint**, it suffice to prove $\int_{x_i}^{x_f} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)\, dx = 0$

However, following the result in (a) (i), we have: $\int_{x_i}^{x_f} \left[\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x)\right] dx = 0$

Comparison gives: $\left[\frac{\partial L}{\partial y'}\eta'(x)\right] \leftrightarrow \left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x)$, which suggests integration by part shall be used.

(a) (iii). As the question setter, I believe that this is the hardest part. To approach the question humanely:

Firstly, observe that our target is to prove $\exists C \in \mathbb{R}$ such that $y'\frac{\partial L}{\partial y'} - L = C$

That is: $y'\frac{\partial L}{\partial y'} - L = C$ is a constant function with variable x

Naturally, you should try to prove that: $\frac{d}{dx}\left(y'\frac{\partial L}{\partial y'} - L\right) = 0$

Secondly, using product rule, it suffice to prove that: $y''(x)\frac{\partial L}{\partial y'} + y'(x)\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] - \frac{dL}{dx} = 0$

At this point, you should spot that: $\frac{dL}{dx}$ is an unknown object. Further clarification is needed !

Coincidentally, '**Hint**: Differentiate $L(x, y(x), y'(x))$ with respect to x by multivariable chain rule'

So you know you are on the correct direction ;)

Thirdly, after computation: $\frac{dL}{dx} = 0 + y'(x)\frac{\partial L}{\partial y} + y''(x)\frac{\partial L}{\partial y'}$

Hence
$$\begin{aligned} y''(x)\frac{\partial L}{\partial y'} + y'(x)\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] - \frac{dL}{dx} &= y''(x)\frac{\partial L}{\partial y'} + y'(x)\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] - \left[y'(x)\frac{\partial L}{\partial y} + y''(x)\frac{\partial L}{\partial y'}\right] \\ &= y'(x)\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - y'(x)\frac{\partial L}{\partial y} \end{aligned}$$

Compare (a) (ii): Multiplying $-y'(x)$ at both sides of $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$ will finish the proof.

(b) (i). This part requires you to compute an integral. However no candidates figured out the substitution $y = c\sin^2\theta$. Actually the substitution is given in an implicit way in (b) (ii). The seemingly sensible substitution is: $y = A(1-\cos\theta)$ and we try to figure value of A by trial and error.

If $A = c$, substitute $y = c(1-\cos\theta)$, denominator becomes $\sqrt{c-y} = \sqrt{c\cos\theta}$

The integral becomes: $\sqrt{2}c\int \frac{1}{\sqrt{\cos\theta}}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$, with non-elementary factor $\frac{1}{\sqrt{\cos\theta}}$: FAIL

If $A = \frac{c}{2}$, substitute $y = \frac{c}{2}(1-\cos\theta)$, $\sqrt{\frac{y}{c-y}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan\frac{\theta}{2}$ (Half Angle Formula for tan)

The integral becomes $\frac{c}{2}\int \tan\frac{\theta}{2}\sin\theta\, d\theta$: DOABLE

Note: $\frac{c}{2}(1-\cos\theta) = c\sin^2\frac{\theta}{2}$, therefore the trial and error result is equivalent to the 'marking scheme'

Of course, $y = c\sin^2\theta$ in the 'marking scheme' is motivated by $c - y = c - c\sin^2\theta = c\cos^2\theta$

But it is a non-standard substitution, so an implicit hint is given

(b) (ii). No comments can be given since no candidate attempted this part.

But picking $L(x, y(x), y'(x)) = \sqrt{\dfrac{1+[y'(x)]^2}{2\mathbf{g}y(x)}}$, $x_i = 0, x_f = a$ scores 1 mark

(b) (iii). Many of you could name the curve as cycloid, but 1 mark does not save you from Q1

Since *Max Shung* decided to hell you to the another side of the Earth

Anyway, it is very interesting to see many of you would rather do Q7, instead of Q1

For obvious reason, Q1 is heavily scaled up with good intention to give you marks

But it turns out: Does More Harm Than Good (?)

And Yes, even though Q7 is non compulsory, more candidates receive non-zero score in Q7 than Q5,

which is the question set by *Max Shung* again! To be honest, *Nelson* and *Max Shung* are hell guys!

Keywords: Calculus of Variation, Euler Lagrange Equation, Beltrami Identity, Lagrangian Mechanics, Compact Support, Vanish at the Boundary, Fundamental Lemma of the Calculus of Variations, Perturbation Method, Action Functional, Least Action Principle.

End of Paper